JOURNAL OF APPROXIMATION THEORY 47, 284–301 (1986)

The Approximate Identity Kernels of Product Type for the Walsh System*

WEIYI SU

Department of Mathematics, Nanjing University, Nanjing, People's Republic of China

Communicated by P. L. Butzer

Received January 4, 1983

At present there are only a few approximate identity kernels for the Walsh system, for example, the p^N-truncated Dirichlet kernel $D_{p^N-1}(t)=\sum_{j=0}^{p^N-1} w_j(t)$ [6]; the Abel–Poisson kernel $\lambda_\gamma(t)=\sum_{k=0}^{\infty}\gamma^k w_k(t)$ [3], and so on. In [6], Zheng has introduced a new kind of approximate identity kernels for the Walsh system—the kernels of product type. In the present paper we discuss the approximation properties of such product type kernels. Estimates of their moments as well as a direct approximation theorem are obtained. Then, to establish an inverse approximation theorem, we need the p-adic derivative of product type kernels and we estimate this derivative in L^1-norm.

In this paper we consider kernels of product type introduced by Zheng (cf. [6]):

$$k_n(t;r;s)=\prod_{j=0}^{n-1}\left(1+a_{j,1}\varphi_j(t)+\cdots+a_{j,p-1}\varphi_j^{p-1}(t)\right),\qquad t\in[0,1),\, n\in\mathcal{N}, \tag{1}$$

where $p\geqslant 2$ is an integer and the coefficients $a_{j,l}$ depend on parameters $s\in\mathcal{N}$, $r\in[0,1)$, that is, $a_{j,l}=a_{j,l}^{(s)}(r)$; $\varphi_j(t)$ are the p-adic Redamacher functions, $\varphi_j(t)=\omega_{p^j}(t)$, $j\in P\equiv\{0,1,2,\ldots\}$; and $\omega_{p^j}(t)$ are the p-adic Walsh functions

$$\omega_m(t)=\exp\frac{2\pi i}{p}(m\odot t),\qquad m\in P,\, t\in[0,1),\, i=\sqrt{-1},$$

where $\odot$ is defined as follows. Let $m=(m_{-\alpha}m_{-\alpha+1}\cdots m_0)$, $\alpha\in P$, and $t=(t_1t_2\cdots)$ be the p-adic expressions of m and t, respectively, m_j, $t_l\in Z\equiv\{0,1,\ldots,p-1\}$, for $j=-\alpha,\ldots,0$; $l=1,2,\ldots$; then

$$m\odot t=\sum_{\mu=1}^{\alpha+1} m_{1-\mu}t_\mu.$$

* For technical reasons unrelated to the author, publication of this paper has been regrettably delayed.

0021-9045/86 $3.00

Obviously, each $k_n(t; r; s)$ is a periodic function with period 1. Zheng [6] has proved that the kernels form an approximate identity. As a matter of fact, he established

THEOREM 1. *Let the kernel $k_n(t; r; s)$ satisfy the following three conditions*:

1. *There is a constant A, independent of n, t, r, s, such that*

$$|k_n(t; r; s)| \leqslant A \min\{(1-r)^{-1}, p^n\}, \qquad t, r \in [0, 1), n, s \in \mathcal{N}.$$

2. *There is a constant B, independent of n, r, s, such that*

$$\max_{1 \leqslant \mu \leqslant p-1} \left|\frac{\eta_j(\omega^\mu; r; s)}{\eta_j(1; r; s)}\right| \leqslant B(1-r)\, p^j, \qquad j \in P,$$

where

$$\eta_j(z; r; s) = 1 + a_{j,1}^{(s)}(r)\, z + \cdots + a_{j,p-1}^{(s)}(r)\, z^{p-1}, \qquad \omega = \exp 2\pi i/p.$$

3. *Assume that for each fixed $j \in P$, $l \in Z(p) - \{0\}$,*

$$\lim_{\substack{s \to \infty \\ r \to 1^-}} a_{j,l}^{(s)}(r) = 1,$$

and that there exist $r' \in [0, 1)$, tending to 1 together with r, and $s' \in \mathcal{N}$, tending to ∞ together with s, such that

$$a_{j+1,l}^{(s)}(r) = a_{j,l}^{(s')}(r'), \qquad j \in P, l \in Z(p) - \{0\}.$$

Then $k_n(t; r; s)$ is an approximate identity kernel. Moreover,

$$\int_0^1 |k_n(t; r; s)|\, dt \leqslant A(p-1) + A \prod_{j=0}^{\infty} (1 + ((p-1)\, B)/p^j).$$

If the coefficients $a_{j,l}$ in (1) depend only on $s \in \mathcal{N}$, $a_{j,l} = a_{j,l}^{(s)}$, we denote the corresponding kernels by $k_n(t; s)$. Zheng [6] also obtained

THEOREM 1′. *Let the kernel $k_n(t; s)$ satisfy the following three conditions*:

1. *There is a constant A′, independent of n, t, s, such that*

$$|k_n(t; s)| \leqslant A' p^n, \qquad s, n \in \mathcal{N}, t \in [0, 1).$$

2. *There is a constant B′, independent of n, s, such that*

$$\max_{1 \leqslant \mu \leqslant p-1} \left|\frac{\eta_j(\omega^\mu; s)}{\eta_j(1; s)}\right| \leqslant B' p^{-s+j}, \qquad j \in \{1, 2, ..., n-1\}, s > n,$$

where $\eta_j(z; s) = 1 + a_{j,1}^{(s)} z + \cdots + a_{j,p-1}^{(s)} z^{p-1}$.

3. *Assume that for each fixed* $j \in P$, $l \in Z(p) - \{0\}$,

$$\lim_{s \to \infty} a_{j,l}^{(s)} = 1,$$

and that there exists $s' \in \mathcal{N}$ *tending to* ∞, *together with* s, *such that*

$$a_{j+1,l}^{(s)} = a_{j,l}^{(s')}, \qquad j \in P, l \in Z(p) - \{0\}.$$

Then $k_n(t; s)$ *is an approximate identity kernel. Moreover,*

$$\int_0^1 |k_n(t; s)|\, dt \leqslant A' \prod_{j=0}^{\infty} (1 + ((p-1) B')/p^j).$$

We give two examples.

EXAMPLE 1. Let $n \in \mathcal{N}$, and

$$a_{j,l}^{(s)}(r) = (1 - (l/p^{s-j}))\, r^{lp^j}, \qquad s \in \mathcal{N}, r \in [0, 1), j \in \{0, 1, ..., n-1\},$$
$$l \in Z(p) - \{0\},$$

where r, s, n satisfy the relation $(1/(1-r))\, p^{n+1} \leqslant p^s$; then the kernel

$$k_n(t; r; s) = \prod_{j=0}^{n-1} \left(1 + \left(1 - \frac{1}{p^{s-j}}\right) r^{p^j} \varphi(t) + \cdots + \left(1 - \frac{p-1}{p^{s-j}}\right) r^{(p-1)p^j} \varphi_j^{p-1}(t)\right.$$

satisfies condition 1 in Theorem 1 with $A = 1$, and condition 2 with $B = 4/\sin(\pi/p)$. Moreover, $\lim_{s \to \infty, r \to 1^-} (1 - l/p^{s-j})\, r^{p^j} = 1$, and $a_{j+1,l}^{(s)}(r) = a_{j,l}^{(s')}(r')$ with $s' = s - 1$ and $r' = r^p$, so condition 3 is also fulfilled. We verify condition 2 as follows. Let $\Delta = \eta_j(\omega^\mu; r; s)/\eta_j(1; r; s)$, then

$$|\Delta| = \left| \frac{1 + \left(1 - \frac{1}{p^{s-j}}\right) r^{p^j} \omega^\mu + \cdots + \left(1 - \frac{p-1}{p^{s-j}}\right) r^{(p-1)p^j} \omega^{(p-1)\mu}}{1 + \left(1 - \frac{1}{p^{s-j}}\right) r^{p^j} + \cdots + \left(1 - \frac{p-1}{p^{s-j}}\right) r^{(p-1)p^j}} \right|$$

$$= \left| \frac{\begin{pmatrix} 1 + r^{p^j} \omega^\mu + \cdots + r^{(p-1)p^j} \omega^{(p-1)\mu} \\ - p^{j-s} r^{p^j} \omega^\mu (1 + 2r^{p^j} \omega^\mu + \cdots + (p-1)\, r^{(p-2)p^j} \omega^{(p-2)\mu}) \end{pmatrix}}{\begin{pmatrix} 1 + r^{p^j} + \cdots + r^{(p-1)p^j} \\ - p^{j-s} r^{p^j} (1 + 2r^{p^j} + \cdots + (p-1)\, r^{p-2)p^j}) \end{pmatrix}} \right|$$

$$\leqslant \frac{\left| \frac{1 - r^{p \cdot p^j}}{1 - r^{p^j} \omega^\mu} \right| + p^{j-s} r^{p^j} (1 + \cdots + p - 1)}{\left| \frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}} - p^{j-s} r^{p^j} (1 + 2r^{p^j} + \cdots + (p-1)\, r^{(p-2)p^j}) \right|} \equiv \frac{\Delta_1}{\Delta_2},$$

say. Since r, s, n satisfy the relation

$$(1/(1-r))\, p^{n+1} \leqslant p^s, \qquad s, n \in \mathcal{N},\ r \in [0, 1),$$

and $|1 - r^{p^j}\omega^\mu| \leqslant 2$, we have for $j = 0, 1, ..., n-1$,

$$\Delta_1 \leqslant \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}\omega^\mu}\right| + \frac{p^{j+2-s}}{2} \leqslant \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}\omega^\mu}\right| + \frac{1-r}{2} \leqslant 2\left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}\omega^\mu}\right|.$$

On the other hand, when $(1/(1-r))\, p^{n+1} \leqslant p^s$, we deduce $2p^n \leqslant p^{s-1}$ by taking $r = \frac{1}{2}$. So

$$\begin{aligned}
\Delta_2 &\geqslant \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}}\right| - p^{j-s} r^{p^j}(1 + 2r^{p^j} + \cdots + (p-1)\, r^{(p-2)p^j}) \\
&\geqslant \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}}\right| - 2p^{j+1-s} \cdot \frac{1}{2}\, (1 + r^{p^j} + \cdots + r^{(p-1)p^j}) \\
&= \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}}\right| - 2p^{j+1-s} \cdot \frac{1}{2} \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}}\right| \geqslant \frac{1}{2} \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}}\right|,
\end{aligned}$$

and

$$|\Delta| \leqslant \frac{\Delta_1}{\Delta_2} \leqslant 4 \left|\frac{1 - r^{p \cdot p^j}}{1 - r^{p^j}\omega^\mu}\right| \leqslant \frac{4}{\sin(\pi/p)}\, (1-r)\, p^j.$$

Hence condition 2 is verified, and

$$\begin{aligned}
k_n(t; r; s) = \prod_{j=0}^{n-1} \Bigg(1 &+ \left(1 - \frac{1}{p^{s-j}}\right) r^{p^j} \varphi_j(t) \\
&+ \cdots + \left(1 - \frac{p-1}{p^{s-j}}\right) r^{(p-1)p^j} \varphi_j^{p-1}(t)\Bigg)
\end{aligned}$$

is an approximate identity kernel.

EXAMPLE 2. Let

$$a_{j,l}^{(s)} = 1 - l/p^{s-j}, \qquad s \in \mathcal{N},\ j \in \{0, 1, ..., n-1\},\ l \in Z(p) - \{0\},$$

where n and s satisfy $n < s$. Then the kernel

$$\begin{aligned}
k_n(t; s) = \prod_{j=0}^{n-1} \Bigg(1 &+ \left(1 - \frac{1}{p^{s-j}}\right) \varphi_j(t) \\
&+ \cdots + \left(1 - \frac{p-1}{p^{s-j}}\right) \varphi_j^{(p-1)}(t)\Bigg)
\end{aligned}$$

satisfies condition 1 in Theorem 1′ with $A'=1$, condition 2 with $B'=1/\sin(\pi/p)$, and condition 3, $\lim_{s\to\infty} a_{j,l}^{(s)}=1$, $a_{j+1,l}^{(s)}=a_{j,l}^{(s')}$ with $s'=s-1$. Therefore $k_n(t;s)$ is an approximate identity kernel by Theorem 1′.

Now we discuss the approximation properties of $k_n(t;r;s)$.

THEOREM 2. *Let $X_{[0,1)}$ be the space $L^q_{[0,1)}$, $1\leqslant q<\infty$, or $WC_{[0,1)}$, where $WC_{[0,1)}$ is the space of all p-adic continuous functions on $[0,1)$ with norm $\|f\|_{WC_{[0,1)}}=\sup_{x\in[0,1)}|f(x)|$; and let*

$$K_n(f;x;r;s)=\int_0^1 k_n(t;r;s)\,f(x\ominus t)\,dt \tag{2}$$

be the singular integral generated by the kernel $k_n(t;r;s)$, where $\ominus$ is the inverse operation of $\oplus$, and

$$x\oplus t=(x_\mu\oplus t_\mu),\qquad x_\mu, t_\mu\in Z(p),\ \mu\in Z,$$

where $x_\mu\oplus t_\mu=x_\mu+t_\mu\ (\mathrm{mod}\ p)$. Then $f\in X_{[0,1)}$ implies

$$\|K_n(f;\cdot;r;s)-f(\cdot)\|_{X_{[0,1)}}\to 0,\qquad s,n\to\infty,\ r\to 1^-.$$

For the moments of the kernels (1), we establish

THEOREM 3. *For the kernel (1) satisfying conditions 1–3 in Theorem 1, the estimations*

$$\text{(i)}\quad \int_0^{1-r} t^\alpha |k_n(t;r;s)|\,dt\leqslant M_1(1-r)^\alpha,\qquad \alpha>0, \tag{3}$$

$$\text{(ii)}\quad \int_{1-r}^1 t^\alpha |k_n(t;r;s)|\,dt\leqslant \begin{cases} M_2(1-r)^\alpha, & 0<\alpha<1, \quad (4)\\ M_3(1-r)\ln\dfrac{1}{1-r}, & \alpha=1, \quad (5)\end{cases}$$

$$\text{(iii)}\quad \int_0^1 t^\alpha |k_n(t;r;s)|\,dt\leqslant M_4(1-r),\qquad \alpha>1, \tag{6}$$

hold, where M_1, M_3 are absolute constants, and M_2, M_4 are constants depending only on α.

Proof. We have by 1

$$\int_0^{1-r} t^\alpha |k_n(t;r;s)|\,dt\leqslant \int_0^{1-r} t^\alpha A\frac{1}{1-r}\,dt = \frac{A}{\alpha+1}(1-r)^\alpha < A(1-r)^\alpha,$$

so that (3) holds.

To prove (4), we note that for any fixed $r \in [0, 1)$ and a given $n \in \mathcal{N}$, there are two cases:

(I) r and n satisfy $0 < 1/p^{n+1} \leqslant 1 - r$. In this case there is $N \in \mathcal{N}$, such that $N - 1 \leqslant n$, and

$$\frac{1}{p^{n+1}} \leqslant \frac{1}{p^N} \leqslant 1 - r < \frac{1}{p^{N-1}}. \tag{7}$$

Obviously, $N \to \infty$ with $r \to 1^-$.

(II) r and n satisfy $0 < 1 - r < 1/p^{n+1}$. We now establish (4) for each of the two cases separately.

For (I), we divide $[0, 1)$ into p^N subintervals

$$I_\mu \equiv I_{\mu,N} \doteq \left[\frac{\mu}{p^N}, \frac{\mu+1}{p^N}\right), \qquad \mu = 0, 1, ..., p^N - 1.$$

Then

$$\begin{aligned}\int_{1-r}^{1} u^\alpha |k_n(u; r; s)| \, du &\leqslant \int_{1/p^N}^{1} u^\alpha |k_n(u; r; s)| \, du \\ &= \left\{\int_{I_1 \cup \cdots \cup I_{p-1}} + \int_{I_p \cup \cdots \cup I_{p^2-1}} + \cdots + \int_{I_{p^{N-1}} \cup \cdots \cup I_{p^N-1}}\right\} \\ &\quad \times u^\alpha |k_n(u; r; s)| \, du.\end{aligned}$$

We estimate $\int_{I_{p^\nu} \cup \cdots \cup I_{p^{\nu+1}-1}}$, $\nu = 0, ..., N-1$. When $\nu = 0$, we may proceed as follows.

If $N < n + 1$, let $t \in I_0$, then $u = t + \mu/p^N \in I_\mu$, $\mu = 1, ..., p-1$, so that

$$\begin{aligned}k_n(u; r; s) &= k_n(t + \mu/p^N; r; s) \\ &= (1 + a_{N-1,1}^{(s)}(r) \, \varphi_{N-1}(t) \, \omega^\mu + \cdots \\ &\quad + a_{N-1,p-1}^{(s)}(r) \, \varphi_{N-1}^{p-1}(t) \, \omega^{\mu(p-1)}) \\ &\quad \times \prod_{\substack{j=0 \\ j \neq N-1}}^{n-1} (1 + a_{j,l}^{(s)}(r) \, \varphi_j(t) + \cdots + a_{j,p-1}^{(s)}(r) \, \varphi_j^{p-1}(t)),\end{aligned}$$

since $\varphi_{N-1}(t) = 1$ for $t \in [0, 1/p^N)$; this implies that

$$k_n(u; r; s) = \frac{\eta_{N-1}(\omega^\mu; r; s)}{\eta_{N-1}(1; r; s)} k_n(t; r; s), \qquad t \in I_0, u \in I_\mu, \mu = 1, ..., p-1.$$

By 2 in Theorem 1, for $t \in I_0$

$$\left|\frac{k_n(t+\mu/p^N; r; s)}{k_n(t; r; s)}\right| = \left|\frac{\eta_{N-1}(\omega^\mu; r; s)}{\eta_{N-1}(1; r; s)}\right| \leqslant B(1-r)\, p^{N-1},$$

and by 1 in Theorem 1

$$|k_n(t+\mu/p^N; r; s)| \leqslant B(1-r)\, p^{N-1} |k_n(t; r; s)| \leqslant ABp^{N-1}.$$

That is, we have for $u \in I_1, \ldots, I_{p-1}$

$$|k_n(u; r; s)| \leqslant ABp^{N-1}. \tag{8}$$

If $N = n+1$, then $k_n(u; r; s) = k_n(t; r; s)$, thus

$$|k_n(u; r; s)| \leqslant Ap^{N-1}, \qquad u \in I_1, \ldots, I_{p-1}.$$

Because the proof for $N = n+1$ is similar to that for $N < n+1$, we only deal with the case for $N < n+1$. Now from (8)

$$\begin{aligned}
\int_{I_\mu} u^\alpha |k_n(u; r; s)|\, du &= \int_{I_0} \left(t+\frac{\mu}{p^N}\right)^\alpha \left|k_n\left(t+\frac{\mu}{p^N}; r; s\right)\right| dt \\
&\leqslant \int_{I_0} \left(t+\frac{\mu}{p^N}\right)^\alpha ABp^{N-1}\, dt \\
&= ABp^{N-1} \frac{1}{\alpha+1} \left\{\left(\frac{\mu+1}{p^N}\right)^{\alpha+1} - \left(\frac{\mu}{p^N}\right)^{\alpha+1}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\int_{I_1 \cup \cdots \cup I_{p-1}} u^\alpha k_n(u; r; s)|\, du &= \sum_{\mu=1}^{p-1} \int_{I_\mu} u^\alpha |k_n(u; r; s)|\, du \\
&\leqslant \frac{AB}{p} \left(\frac{1}{p^N}\right)^\alpha \sum_{\mu=1}^{p-1} \frac{1}{\alpha+1} [(\mu+1)^{\alpha+1} - \mu^{\alpha+1}] \\
&= \frac{AB}{p} \left(\frac{1}{p^N}\right)^\alpha \sum_{\mu=1}^{p-1} (\mu+\xi)^\alpha \\
&\leqslant \frac{AB}{p} \left(\frac{1}{p^N}\right)^\alpha D, \qquad \xi \in (0, 1),
\end{aligned}$$

where $D = \sum_{\mu=1}^{p-1} (\mu+1)^\alpha$, and the mean value theorem was used; therefore

$$\int_{I_1 \cup \cdots \cup I_{p-1}} u^\alpha |k_n(u; r; s)|\, du \leqslant \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha. \tag{9}$$

Now we estimate $\int_{I_1 \cup \cdots \cup I_{p^2-1}}$. Since

$$\int_{I_p \cup \cdots \cup I_{p^2-1}} u^\alpha |k_n(u; r; s)| \, du = \left\{ \sum_{\mu=1}^{p-1} \int_{I_{\mu p}} + \sum_{\mu=1}^{p-1} \int_{I_{\mu p+1} \cup \cdots \cup I_{(\mu+1)p-1}} \right\} u^\alpha |k_n(u; r; s)| \, du,$$

and the mappings $t \to u = t + \mu/p^{N-1}$, $\mu = 1,..., p-1$, transform the interval I_0 onto $I_{\mu p}$, respectively, this yields

$$|k_n(t + \mu/p^{N-1}; r; s)| \leqslant B(1-r)\, p^{N-2} |k_n(t; r; s)|, \qquad t \in I_0,$$

thus

$$\begin{aligned}
\sum_{\mu=1}^{p-1} \int_{I_{\mu p}} & u^\alpha |k_n(u; r; s)| \, du \\
&= \sum_{\mu=1}^{p-1} \int_{I_0} \left(t + \frac{\mu}{p^{N-1}} \right)^\alpha \left| k_n \left(t + \frac{\mu}{p^{N-1}}; r; s \right) \right| dt \\
&\leqslant \sum_{\mu=1}^{p-1} ABp^{N-2} \int_{I_0} \left(t + \frac{\mu}{p^{N-1}} \right)^\alpha dt \\
&= \frac{AB}{p^2} \left(\frac{1}{p^N} \right)^\alpha \sum_{\mu=1}^{p-1} (\mu p + \xi_1)^\alpha \leqslant \frac{ABD}{p^{2-\alpha}} \left(\frac{1}{p^N} \right)^\alpha,
\end{aligned}$$

that is,

$$\sum_{\mu=1}^{p-1} \int_{I_{\mu p}} u^\alpha |k_n(u; r; s)| \leqslant \frac{ABD}{p} \frac{1}{p^{1-N}} \left(\frac{1}{p^N} \right)^\alpha. \tag{10}$$

For $\sum_{\mu=1}^{p-1} \int_{I_{\mu p+1} \cup \cdots \cup I_{(\mu+1)p-1}}$, the same mappings $t \to u = t + \mu/p^{N-1}$ are used, and they transform I_ν onto $I_{\mu p+\nu}$, $\mu, \nu = 1,..., p-1$. By conditions 1 and 2 in Theorem 1

$$\begin{aligned}
\sum_{\mu=1}^{p-1} \int_{I_{\mu p+1} \cup \cdots \cup I_{(\mu+1)p-1}} & u^\alpha |k_n(u; r; s)| \, du \\
&\leqslant \sum_{\mu=1}^{p-1} B(1-r)\, p^{N-2} \int_{I_1 \cup \cdots \cup I_{p-1}} \left(t + \frac{\mu}{p^{N-1}} \right)^\alpha |k_n(t; r; s)| \, dt \\
&\leqslant \sum_{\mu=1}^{p-1} B(1-r)\, p^{N-2} \sum_{\mu_1=1}^{p-1} \int_{I_0} \left(x + \frac{\mu}{p^{N-1}} + \frac{\mu_1}{p^N} \right)^\alpha \\
&\quad \times B(1-r)\, p^{N-1} |k_n(x; r; s)| \, dx
\end{aligned}$$

$$\leqslant \frac{AB^2}{p^2}\left(\frac{1}{p^N}\right)^\alpha \sum_{\mu=1}^{p-1} \sum_{\mu_1=1}^{p-1} \frac{1}{\alpha+1}\left[\mu p+\mu_1+1)^{\alpha+1}-(\mu p+\mu_1)^{\alpha+1}\right]$$
$$=\frac{AB^2}{p^2}\left(\frac{1}{p^N}\right)^\alpha \sum_{\mu=1}^{p-1} \frac{1}{\alpha+1}\left[\mu p+p)^{\alpha+1}-(\mu p+1)^{\alpha+1}\right]$$
$$=\frac{AB^2}{p^2}\left(\frac{1}{p^N}\right)^\alpha \sum_{\mu=1}^{p-1} (\mu p+1+\eta)^\alpha (p-1), \qquad \eta\in(0, p-1).$$

Now under the same notations

$$\sum_{\mu=1}^{p-1} \int_{I_{\mu p+1}\cup\cdots\cup I_{(\mu+1)p-1}} u^\alpha |k_n(u;r;s)|\,du \leqslant \frac{AB^2D}{p^{1-\alpha}}\left(\frac{1}{p^N}\right)^\alpha \frac{p-1}{p}. \tag{11}$$

Combining (10) and (11), we have

$$\int_{I_p\cup\cdots\cup I_{p^2-1}} u^\alpha |k_n(u;r;s)|\,du \leqslant \frac{ABD}{p}\frac{1}{p^{1-\alpha}}\left(\frac{1}{p^N}\right)^\alpha (1+(p-1)B). \tag{12}$$

Then in virtue of (9) and (12)

$$\int_{I_1\cup\cdots\cup I_{p^2-1}} u^\alpha |k_n(u;r;s)|\,du \leqslant \frac{ABD}{p}\left(\frac{1}{p^N}\right)^\alpha \left(1+\frac{(p-1)B}{p^0}\right)\left(1+\frac{1}{p^{1-\alpha}}\right).$$

For general v, $v=1, 2,..., N-1$, we estimate $\int_{I_{p^v}\cup\cdots\cup I_{p^{v+1}-1}}$ by induction. Suppose that for $v\geqslant 1$

$$\int_{I_{p^{v-1}}\cup\cdots\cup I_{p^v-1}} u^\alpha |k_n(u;r;s)|\,du$$
$$\leqslant \frac{ABD}{p}\frac{1}{p^{(v-1)(1-\alpha)}}\left(\frac{1}{p^N}\right)^\alpha \left(1+\frac{(p-1)B}{p^0}\right)\cdots\left(1+\frac{(p-1)B}{p^{v-2}}\right), \tag{13}$$

we write

$$\int_{I_{p^v}\cup\cdots\cup I_{p^{v+1}-1}} u^\alpha |k_n(u;r;s)|\,du$$
$$=\sum_{\mu=1}^{p-1}\left\{\int_{I_{\mu p^v}}+\int_{I_{\mu p^v+1}\cup\cdots\cup I_{(\mu+1)p^v-1}}\right\} u^\alpha |k_n(u;r;s)|\,du.$$

The mappings $t\to u=t+\mu/p^{N-v}$, $\mu=1,..., p-1$, transform I_0 onto $I_{\mu p^v}$, respectively, and I_{μ_1} onto $I_{\mu p^v+\mu_1}$, $\mu_1=1,..., p-1$. Hence by conditions 1 and 2 in Theorem 1,

$$\left|k_n\left(t+\frac{\mu}{p^{N-\nu}};r;s\right)\right| = \left|\frac{\eta_{N-\nu-1}(\omega^{\mu};r;s)}{\eta_{N-\nu-1}(1;r;s)}\right| |k_n(t;r;s)|$$
$$\leqslant B(1-r)\, p^{N-\nu-1} |k_n(t;r;s)|, \qquad t \in I_0,$$
$$\left|k_n\left(t+\frac{\mu_1}{p^{N-\nu}};r;s\right)\right| = \left|\frac{\eta_{N-\nu-1}(\omega^{\mu_1};r;s)}{\eta_{N-\nu-1}(1;r;s)}\right| |k_n(t;r;s)|, \qquad t \in I_{\mu_1}.$$

These imply by induction that

$$\int_{I_{p^\nu} \cup \cdots \cup I_{p^{\nu+1}-1}} u^\alpha |k_n(u;r;s)|\, du$$
$$\leqslant \frac{ABD}{p} \frac{1}{p^{\nu(1-\alpha)}} \left(\frac{1}{p^N}\right)^\alpha \left(1+\frac{(p-1)\,B}{p^0}\right) \cdots \left(1+\frac{(p-1)\,B}{p^{\nu-1}}\right),$$

and

$$\int_{I_1 \cup \cdots \cup I_{p^{\nu+1}-1}} u^\alpha |k_n(u;r;s)|\, du$$
$$\leqslant \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha \left(1+\frac{1}{p^{1-\alpha}}+\cdots+\frac{1}{p^{\nu(1-\alpha)}}\right) \prod_{j=0}^{\nu-1} \left(1+\frac{(p-1)\,B}{p^j}\right). \tag{14}$$

Setting $\nu = N-1$ in (14) we obtain

$$\int_{I_1 \cup \cdots \cup I_{p^N-1}} u^\alpha |k_n(u;r;s)|\, du$$
$$\leqslant \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha \left(1+\frac{1}{p^{1-\alpha}}+\cdots+\frac{1}{p^{(N-1)(1-\alpha)}}\right) \prod_{j=0}^{N-2} \left(1+\frac{(p-1)\,B}{p^j}\right).$$

Since $0<\alpha<1$, the sum $1+1/p^{1-\alpha}+\cdots+1/p^{(N-1)(1-\alpha)}$ is bounded with $p^{1-\alpha}(p^{1-\alpha}-1)^{-1}$, so it follows from (7) that

$$\int_{I_1 \cup \cdots \cup I_{p^N-1}} u^\alpha |k_n(u;r;s)|\, du$$
$$\leqslant \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha \frac{p^{1-\alpha}}{p^{1-\alpha}-1} \prod_{j=0}^{\infty} \left(1+\frac{(p-1)\,B}{p^j}\right) \leqslant M_2(1-r)^\alpha,$$

where M_2 is a constant depending only on α. Equation (4) is proved for Case (I).

For (II), note that there exists $m \in \mathcal{N}$ such that $m>n$ and

$$0<\frac{1}{p^{m+1}}<1-r\leqslant\frac{1}{p^m}<\frac{1}{p^{n+1}}, \tag{15}$$

we let $m \to \infty$ together with $r \to 1^-$. By the same method used above, (6) may also be obtained. By the way, when $\alpha > 1$ formula (14) still holds; it follows from (14) that

$$
\begin{aligned}
&\int_0^1 u^\alpha |k_n(u; r; s)| \, du \\
&= \left\{ \int_{I_0} + \int_{I_1 \cup \cdots \cup I_{p^N - 1}} \right\} u^\alpha |k_n(u; r; s)| \, du \\
&\leqslant \frac{A}{\alpha+1} \left(\frac{1}{p^N}\right)^\alpha + \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha (1 + p^{\alpha-1} + \cdots + p^{(N-1)(\alpha-1)}) \\
&\quad \times \prod_{j=0}^{N-2} \left(1 + \frac{(p-1)\,B}{p^j}\right) \\
&\leqslant \frac{A}{\alpha+1} \left(\frac{1}{p^N}\right)^\alpha + \frac{ABD}{p} \left(\frac{1}{p^N}\right)^\alpha \frac{(p^{\alpha-1})^N - 1}{p^{\alpha-1} - 1} \prod_{j=0}^{\infty} \left(1 + \frac{(p-1)\,B}{p^j}\right) \\
&\leqslant \frac{A}{\alpha+1} \left(\frac{1}{p^N}\right)^\alpha + \frac{ABD}{p(p^{\alpha-1} - 1)} \left(\frac{1}{p^N}\right)^\alpha \prod_{j=0}^{\infty} \left(1 + \frac{(p-1)\,B}{p^j}\right) \\
&\leqslant M_4(1-r),
\end{aligned}
$$

where M_4 is a constant depending only on α; hence (6) is valid.

We turn to estimate $\int_{1-r}^1 u |k_n(u; r; s)| \, du$. Repeating the steps used above to deduce, e.g., for $N < n+1$, we have

$$
\int_{I_1 \cup \cdots \cup I_{p-1}} u |k_n(u; r; s)| \, du \leqslant \frac{ABD_0}{p} \frac{1}{p^N},
$$

$$
\int_{I_1 \cup \cdots \cup I_{p^2-1}} u |k_n(u; r; s)| \, du \leqslant \frac{ABD_0}{p} \frac{1}{p^N} \left(1 + \frac{(p-1)\,B}{p^0}\right),
$$

where $D_0 \equiv \sum_{\mu=1}^{p-1} (\mu + 1)$, and generally,

$$
\int_{I_1 \cup \cdots \cup I_{p^\nu - 1}} u |k_n(u; r; s)| \, du \leqslant \frac{ABD_0}{p} \frac{1}{p^N} \cdot \nu \cdot \prod_{j=0}^{\infty} \left(1 + \frac{(p-1)\,B}{p^j}\right).
$$

Let $\nu = N$ for (I). We have by (7), $N - 1 \leqslant (\ln(1/(1-r)))/(\ln p)$, thus

$$
\int_{1-r}^1 u |k_n(u; r; s)| \, du \leqslant M_3(1-r) \ln(1/(1-r)), \qquad M_3 = \text{const}.
$$

For (II), the result (5) follows similarly, and the proof is complete.

For the Walsh system, Lipschitz classes and the modulus of continuity are defined respectively as follows:

$$\mathrm{Lip}(\alpha; X_{[0,1)}) = \{f \in X_{[0,1)} : \|f(\cdot \oplus h) - f(\cdot)\|_{X_{[0,1)}} \leqslant Mh^{\alpha}\}, \qquad \alpha > 0,$$

$$\omega(f; \delta; X_{[0,1)}) = \sup_{0 \leqslant h \leqslant \delta} \|f(\cdot \oplus h) - f(\cdot)\|_{X_{[0,1)}}.$$

In virtue of Theorem 3, the direct approximation theorem follows immediately.

THEOREM 4. *Let* $f \in \mathrm{Lip}(\alpha; X_{[0,1)})$, $\alpha > 0$. *Then for* $r \to 1^-$, $n \to \infty$, $s \to \infty$,

$$\|K_n(f; \cdot; r; s) - f(\cdot)\|_{X_{[0,1)}} = \begin{cases} O((1-r)^{\alpha}), & 0 < \alpha < 1, \\ O\left((1-r) \ln \dfrac{1}{1-r}\right), & \alpha = 1, \\ O(1-r), & \alpha > 1. \end{cases} \tag{16}$$

Proof. It follows from Minkowski's inequality that

$$\begin{aligned} \|K_n(f; \cdot; r; s) - f(\cdot)\|_{X_{[0,1)}} &= \left\| \int_0^1 [f(\cdot \ominus u) - f(\cdot)]\, k_n(u; r; s)\, du \right\|_{X_{[0,1)}} \\ &\leqslant \int_0^1 \|f(\cdot \oplus u) - f(\cdot)\|_{X_{[0,1)}} |k_n(u; r; s)|\, du \\ &\leqslant \int_0^1 \omega(f; u; X_{[0,1)})\, |k_n(u; r; s)|\, du. \end{aligned}$$

But $f \in \mathrm{Lip}(\alpha; X_{[0,1)})$ implies $\omega(f; u; X_{[0,1)}) \leqslant Mu^{\alpha}$, $\alpha > 0$; thus (16) follows by Theorem 3.

To prove the inverse approximation theorem for the singular integral $K_n(f; x; r; s)$, we seek the p-adic derivative [2, 5] of the kernel $k_n(t; r; s)$.

DEFINITION. If $f(x)$ is a real or complex-valued function on $[0, 1)$, then its p-adic derivative (pointwise) is defined as [5]

$$f^{\langle 1 \rangle}(x) = \lim_{N \to \infty} \sum_{l=0}^{N} p^l \sum_{\mu=0}^{p-1} A_{\mu} f(x \oplus \mu p^{-l-1}), \qquad x \in [0, 1),$$

whenever the limit exists, where $A_0 = (p-1)/2$, $A_{\mu} = \omega^{\mu}/(1 - \omega^{\mu})$, $\mu = 1, \ldots, p-1$.

Let $t = (t_1 t_2 \cdots)$ be the p-adic expression of $t \in [0, 1)$; then for $N > n$ (note that $\sum_{\mu=0}^{p-1} A_{\mu} = 0$)

$$\begin{aligned} J_N &\equiv \sum_{l=0}^{N} p^l \sum_{\mu=0}^{p-1} A_{\mu} k_n(t \oplus \mu p^{-l-1}; r; s) \\ &= \sum_{l=0}^{n-1} p^l \left\{ \prod_{\substack{j=0 \\ j \neq l}}^{n-1} (1 + a_{j,1}^{(s)}(r)\, \omega^{t_{j+1}} + \cdots + a_{j}^{(s)}{}_{-1}(r)\, \omega^{(p-1)t_{j+1}}) \right\} \end{aligned}$$

$$\times\left\{\sum_{\mu=0}^{p-1} A_\mu(1+a_{l,1}^{(s)}(r)\,\omega^{\mu+t_{l+1}}+\cdots+a_{l,p-1}^{(s)}(r)\,\omega^{\mu+(p-1)t_{l+1}})\right\}$$

$$+\sum_{l=n}^{N} p^l\left\{\prod_{j=0}^{n-1}(1+a_{j,1}^{(s)}(r)\,\omega^{t_{j+1}}+\cdots+a_{j,p-1}^{(s)}(r)\,\omega^{(p-1)t_{j+1}})\right\}$$

$$\cdot\left(\sum_{\mu=0}^{p-1} A_\mu\right)$$

$$=\sum_{l=0}^{n-1} p^l\left\{\prod_{\substack{j=0\\ j\neq l}}^{n-1}\eta_j(\omega^{t_{j+1}};r;s)\right\}\left\{\sum_{\mu=0}^{p-1}A_\mu\eta_l(\omega^{\mu+t_{j+1}};r;s)\right\}$$

$$=\eta_0(1;r;s)\cdots\eta_{n-1}(1;r;s)$$

$$\times\left\{\sum_{l=0}^{n-1} p^l\prod_{\substack{j=0\\ j\neq l}}^{N-1}\frac{\eta_j(\omega^{t_{j+1}};r;s)}{\eta_j(1;r;s)}\left[\sum_{\mu=0}^{p-1}A_\mu\frac{\eta_l(\omega^{\mu+t_{l+1}};r;s)}{\eta_l(1;r;s)}\right]\right\}.$$

The limit of J_N exists as $N\to\infty$, and we have

$$k_n^{\langle 1\rangle}(t;r;s)=\lim_{N\to\infty}J_N$$

$$=k_n\left(\frac{1}{p^{n+1}};r;s\right)\left\{\sum_{l=0}^{n-1}p^l\left[\prod_{\substack{j=0\\ j\neq l}}^{n-1}\frac{\eta_j(\omega^{t_{j+1}};r;s)}{\eta_j(1;r;s)}\right.\right.$$

$$\left.\left.\times\left(\sum_{\mu=0}^{p-1}A_\mu\frac{\eta_l(\omega^{\mu+t_{l+1}};r;s)}{\eta_l(1;r;s)}\right)\right]\right\};$$

thus by conditions 1 and 2 in Theorem 1

$$|k_n^{\langle 1\rangle}(t;r;s)|\leqslant\left|k_n\left(\frac{1}{p^{n+1}};r;s\right)\right|\left\{\sum_{l=0}^{n-1}p^l\left[\prod_{\substack{j=0\\ j\neq l}}^{n-1}\left|\frac{\eta_j(\omega^{t_{j+1}};r;s)}{\eta_j(1;r;s)}\right|\right.\right.$$

$$\left.\left.\times\left(\sum_{\mu=0}^{p-1}|A_\mu|\left|\frac{\eta_l(\omega^{u+t_{l+1}};r;s)}{\eta_l(1;r;s)}\right|\right)\right]\right\}$$

$$\leqslant\frac{A}{1-r}\sum_{l=0}^{n-1}p^lB^{n-1}(1-r)^{n-1}$$

$$\times\left\{p^{0+1+\cdots+l-1+l+1+\cdots+n-1}\cdot B\cdot(1-r)\cdot p^l\cdot\left(\sum_{\mu=0}^{p-1}|A_\mu|\right)\right\}$$

$$=\frac{Ac}{p-1}\frac{1}{1-r}B^n(1-r)^n\,p^{(n^2+n)/2},$$

where $c=\sum_{\mu=0}^{p-1}|A_\mu|$. From this estimation of $k_n^{\langle 1\rangle}(t;r;s)$ we see that if

$p^n(1-r)-1=O(1/n)$, we have $\ln p^n(1-r)=\ln(1+O(1/n))=O(1/n)$; then $n\ln p^n(1-r)=O(1)$, therefore $n\{\ln B+\ln(1-r)+((n+1)/2)\ln p\}=O(1/n)$ and $(B(1-r)\,p^{(n+1)/2})^n=O(1)$; so the following theorem is obtained immediately.

THEOREM 5. *For the p-adic derivative of $k_n(t;r;s)$ we have*

$$\|k_n^{\langle 1\rangle}(\cdot\,;r;s)\|_{L[0,1)}=O(1/(1-r))$$

for $r\to 1^-$, $n\to\infty$, $s\to\infty$ with the relation $p^n(1-r)-1=O(1/n)$.

We now prove the inverse approximation theorem.

THEOREM 6. *Let $0<\alpha<1$, $f\in X_{[0,1)}$. If for $n\to\infty$ with $r=1-1/p^n$,*

$$\|K_n(f;\cdot\,;r;s)-f(\cdot)\|_{X_{[0,1)}}=O(1/p^{n\alpha}), \tag{17}$$

then $f\in \operatorname{Lip}(\alpha;X_{[0,1)})$.

Proof. Let $\delta>0$ be given. From (17) there exists a constant $G>0$ such that

$$\left\|K_n\left(f;\cdot\,;1-\frac{1}{p^n};s\right)-f(\cdot)\right\|_{X_{[0,1)}}\leqslant G\left(\frac{1}{p^n}\right)^{\alpha},\qquad n\in\mathcal{N}. \tag{18}$$

Setting $r_n=1-1/p^n$, $n=2,3,\ldots$, we construct the sequence

$$\bar U_2(t)=K_2(f;t;r_2;s)=\int_0^1 k_2(t\ominus u;r_2;s)\,f(u)\,du, \tag{19}$$

$$\begin{aligned}\bar U_n(t)&=K_n(f;t;r_n;s)-K_{n-1}(f;t;r_{n-1};s)\\&=\int_0^1[k_n(t\ominus u;r_n;s)-k_{n-1}(t\ominus u;r_{n-1};s)]\,f(u)\,du.\end{aligned} \tag{20}$$

thus by (18)

$$\begin{aligned}\|\bar U_n\|_{X_{[0,1)}}&\leqslant\|K_n(f;\cdot\,;r_n;s)-f(\cdot)\|_{X_{[0,1)}}\\&\quad+\|K_{n-1}(f;\cdot\,;r_{n-1};s)-f(\cdot)\|_{X_{[0,1)}}\\&\leqslant G\left(\frac{1}{p^{n\alpha}}+\frac{1}{p^{(n-1)\alpha}}\right).\end{aligned} \tag{21}$$

Then $\|\bar U_n\|_{X_{[0,1)}}$ is bounded with $G(1+p)$, and

$$\sum_{n=2}^{l}\bar U_n(t)=K_l(f;t;r_l;s).$$

On the other hand, since $k_n(t; r; s)$ is an approximate identity kernel,

$$\lim_{l\to\infty}\left\|\sum_{n=2}^{l}\bar{U}_n(\cdot)-f(\cdot)\right\|_{X[0,1)}=0, \tag{22}$$

$$\lim_{l\to\infty}\left\|\sum_{n=2}^{l}\bar{U}_n(\cdot\oplus h)-f(\cdot\oplus h)\right\|_{X[0,1)}=0, \qquad h\in[0,1), \tag{23}$$

and consequently

$$\|f\|_{X[0,1)}\leqslant\sum_{n=2}^{\infty}\|U_n\|_{X[0,1)}. \tag{24}$$

Relations (22) and (23) imply that for $h\in[0,1)$

$$\lim_{l\to\infty}\left\|\left\{\sum_{n=2}^{l}\bar{U}_n(\cdot\oplus h)-\bar{U}_n(\cdot)\right\}-\{f(\cdot\oplus h)-f(\cdot)\}\right\|_{X[0,1)}=0. \tag{25}$$

It follows from (24) and (25) that for any integer $l\geqslant 2$ and $h\in[0,1)$

$$\begin{aligned}\|f(\cdot\oplus h)-f(\cdot)\|_{X[0,1)}&\leqslant\sum_{n=2}^{\infty}\|\bar{U}_n(\cdot\oplus h)-\bar{U}_n(\cdot)\|_{X[0,1)}\\&\leqslant\sum_{n=2}^{l}\|\bar{U}_n(\cdot\oplus h)-\bar{U}_n(\cdot)\|_{X[0,1)}\\&\quad+2\sum_{n=l+1}^{\infty}\|\bar{U}_n\|_{X[0,1)}\equiv J_1+J_2,\end{aligned} \tag{26}$$

say. For $h\in[0,\delta)$, there is $s\in\mathcal{N}$ such that

$$p^{-s}\leqslant h<p^{-s+1}. \tag{27}$$

J_1 is estimated as follows. Using the Walsh–Fourier coefficient method [1], we deduce

$$\bar{U}_n(x\oplus h)-\bar{U}_n(x)=\bar{W}_1^{(s)}(x)\circledast[\bar{U}_n^{\langle 1\rangle}(x\oplus h)-\bar{U}_n^{\langle 1\rangle}(x)],$$

where the Walsh–Fourier coefficients of $\bar{W}_1^{(s)}(x)$ are

$$[\bar{W}_1^{(s)}(\cdot)]^\wedge(j)=\int_0^1\bar{W}_1^{(s)}(y)\,\bar{\omega}_j(y)\,dy=\begin{cases}0, & 0\leqslant j<p^{s-1}, s\in\mathcal{N},\\ 1/j, & p^{s-1}\leqslant j, s\in\mathcal{N},\end{cases}$$

and $\circledast$ is the p-adic convolution of two functions f and g,

$$(f\circledast g)(x)=\int_0^1 f(x\ominus u)\,g(u)\,du.$$

In view of $\|\bar{W}_1^{(s)}\|_{L^1_{[0,1)}} = O(p^{-s})$, $s \to \infty$ (see [2, 4]) there is a constant $H > 0$ such that $\|\bar{W}_1^{(s)}\|_{L^1_{[0,1)}} \leqslant (H/2)\, p^{-s}$, $s \in \mathcal{N}$. This yields by (21)

$$\begin{aligned}\|\bar{U}_n(\cdot \oplus h) - \bar{U}_n(\cdot)\|_{X_{[0,1)}} &\leqslant \|\bar{W}_1^{(s)}\|_{L^1_{[0,1)}} \|\bar{U}_n^{\langle 1\rangle}(\cdot \oplus h) - \bar{U}_n^{\langle 1\rangle}(\cdot)\|_{X_{[0,1)}} \\ &\leqslant \frac{H}{2} p^{-s} \cdot 2 \|\bar{U}_n^{\langle 1\rangle}\|_{X_{[0,1)}} \\ &= Hp^{-s} \|\bar{U}_n^{\langle 1\rangle}\|_{X_{[0,1)}}, \qquad p^{-s} \leqslant h < p^{-s+1}. \end{aligned} \tag{28}$$

To estimate $\|\bar{U}_n^{\langle 1\rangle}\|_{X_{[0,1)}}$, note that the equality

$$K_n(K_{n-1}(f; t; r_{n-1}; s); t; r_n; s) = K_{n-1}(K_n(f; t; r_n; s); t; r_{n-1}; s)$$

is valid for each $n \in \mathcal{N}$. Thus by (20)

$$\begin{aligned}\bar{U}_n(t) = K_n(f; t; r_n; s) &- K_n(K_{n-1}(f; x; r_{n-1}; s); t; r_n; s) \\ &+ K_{n-1}(K_n(f; x; r_n; s); t; t_{n-1}; s) - K_{n-1}(f; t; r_{n-1}; s),\end{aligned}$$

and this implies

$$\begin{aligned}\bar{U}_n^{\langle 1\rangle}(t) = \int_0^1 & k_n^{\langle 1\rangle}(t \ominus u; r_n; s) \{f(u) - K_{n-1}(f; u; r_{n-1}; s)\}\, du \\ &- \int_0^1 k_{n-1}^{\langle 1\rangle}(t \ominus u; r_{n-1}; s) \{f(n) - K_n(f; u; r_n; s)\}\, du,\end{aligned}$$

whence by (19) and Theorem 5

$$\begin{aligned}\|\bar{U}_n^{\langle 1\rangle}\|_{X_{[0,1)}} &\leqslant \|k_n^{\langle 1\rangle}(\cdot; r_n; s)\|_{L^1_{[0,1)}} \|K_{n-1}(f; \cdot; r_{n-1}; s) - f(\cdot)\|_{X_{[0,1)}} \\ &\quad + \|k_{n-1}^{\langle 1\rangle}(\cdot; r_{n-1}; s)\|_{L^1_{[0,1)}} \|K_n(f; \cdot; r_n; s) - f(\cdot)\|_{X_{[0,1)}} \\ &\leqslant G(1 - r_{n-1})^\alpha \frac{M}{1 - r_n} + G(1 - r_n)^\alpha \frac{M}{1 - r_{n-1}} \\ &= GM(p^\alpha + p^{-1})\, p^{n(1-\alpha)}. \end{aligned} \tag{29}$$

Choosing an integer $l \geqslant 2$ such that $l - 1 \leqslant s < l$ we have by (26), (28), (29), (21),

$$\begin{aligned}&\|f(\cdot \oplus h) - f(\cdot)\|_{X_{[0,1)}} \\ &\quad\leqslant \sum_{n=2}^{l} Hp^{-s} \|\bar{U}_n^{\langle 1\rangle}\|_{X_{[0,1)}} + 2 \sum_{n=l+1}^{\infty} G\left(\frac{1}{p^{n\alpha}} + \frac{1}{p^{(n-1)\alpha}}\right) \\ &\quad\leqslant HGMp^{-s}(p^\alpha + p^{-1}) \sum_{n=2}^{l} p^{n(1-\alpha)} + 2G(1 + p^\alpha) \sum_{n=l+1}^{\infty} p^{-n\alpha} \\ &\quad\leqslant 2G(1 + p^\alpha) \left\{HMp^{-s} \sum_{n=2}^{l} p^{n(1-\alpha)} + \frac{p^{-\alpha}}{1 - p^{-\alpha}}\, p^{-s\alpha}\right\}. \end{aligned} \tag{30}$$

And since $0<\alpha<1$, $p^{l-1}\leqslant p^s<p^l$, it follows that

$$p^{-s}\sum_{n=2}^{l} p^{-n(1-\alpha)}\leqslant p^{-s}\frac{p^{(l+1)(1-\alpha)}}{p^{1-\alpha}-1}\leqslant p^{-s}\cdot p^{s+2}\frac{p^{-\alpha}\cdot p^{-l\alpha}}{p^{1-\alpha}-1}$$
$$\leqslant\frac{p^{2-\alpha}}{p^{1-\alpha}-1}\cdot p^{-s\alpha}. \tag{31}$$

Therefore, by (30) and (31)

$$\|f(\cdot\oplus h)-f(\cdot)\|_{X_{[0,1)}}\leqslant 2G(1+p^{\alpha})\{HML_1\,p^{-s\alpha}+L_2\,p^{-s\alpha}\}\equiv G_0\,p^{-s\alpha},$$

where L_1, L_2, and G_0 are constants depending only on α. Hence, by (27)

$$\|f(\cdot\oplus h)-f(\cdot)\|_{X_{[0,1)}}\leqslant G_0\delta^{\alpha},\qquad 0<\alpha<1.$$

This yields $\sup_{0\leqslant h<\delta}\|f(\cdot\oplus h)-f(\cdot)\|_{X_{[0,1)}}\leqslant G_0\delta^{\alpha}$, which completes the proof.

COROLLARY. *Let $f\in X_{[0,1)}$ and $0<\alpha<1$. Then*

$$\|K_n(f;\cdot;r;s)-f(\cdot)\|_{X_{[0,1)}}=O((1-r)^{\alpha}),\qquad r\to 1^-,\ n\to\infty,$$

if and only if $f\in \mathrm{Lip}(\alpha;X_{[0,1)})$.

For the kernel $k_n(t;s)$ with conditions 1–3 in Theorem 1′, theorems corresponding to Theorems 2–6 are proved to be valid; we denote them by Theorems 2′–6′, respectively. Here we only state Theorems 3′–6′ but omit their proofs since they are rather similar to the previous ones.

THEOREM 3′. *For the kernel $k_n(t;s)$ satisfying conditions 1–3 in Theorem 1′, we have*

(i) $\int_0^{1/p^n} t^{\alpha}|k_n(t;s)|\,dt\leqslant M'(1/p^n)^{\alpha}$, $\qquad\alpha>0$,

(ii) $\int_{1/p^n}^{1} t^{\alpha}|k_n(t;s)|\,dt\leqslant\begin{cases}M'_2(1/p^n)^{\alpha}, & 0<\alpha<1,\\ M'_3(1/p^n)\ln p^n, & \alpha=1,\end{cases}$

(iii) $\int_0^1 t^{\alpha}|k_n(t;s)|\,dt\leqslant M'_4(1/p^n)$, $\qquad\alpha>1$,

where M'_1, M'_3 are absolute constants, and M'_2, M'_4 are constants depending only on α.

THEOREM 4′. *If $f\in\mathrm{Lip}(\alpha;X_{[0,1)})$, then for $n,s\to\infty$ we have*

$$\|K_n(f;\cdot;s)-f(\cdot)\|_{X_{[0,1)}}=\begin{cases}O((1/p^n)^{\alpha}), & 0<\alpha<1,\\ O((1/p^n)\ln p^n), & \alpha=1,\\ O(1/p^n), & \alpha>1.\end{cases}$$

THEOREM 5′. *The p-adic derivative of the kernel* $k_n(t; s)$ *has the expression*

$$k_n^{\langle 1 \rangle}(t; s) = k_n\left(\frac{1}{p^{n+1}}; s\right) \left\{ \sum_{l=0}^{n-1} p^l \left[\prod_{\substack{j=0 \\ j \neq l}}^{n-1} \frac{\eta_j(\omega^{t_{j+1}}; s)}{\eta_j(1; s)} \times \left(\sum_{\mu=0}^{p-1} A_\mu \frac{\eta_l(\omega^{\mu + t_{l+1}}; s)}{\eta_l(1; s)} \right) \right] \right\}$$

and it satisfies the estimate

$$\|k_n^{\langle 1 \rangle}(\cdot; s)\|_{L^1_{[0,1)}} = O(p^n), \qquad n \to \infty.$$

THEOREM 6′. *Let* $0 < \alpha < 1, f \in X_{[0,1)}$. *If*

$$\|K_n(f; \cdot; s) - f(\cdot)\|_{X_{[0,1)}} = O((1/p^n)^\alpha), \qquad n, s \to \infty,$$

then $f \in \mathrm{Lip}(\alpha; X_{[0,1)})$.

ACKNOWLEDGMENT

The author could like to thank Professor Wei-xing Zheng heartily for his encouragement and help. Many thanks also to Professor P. L. Butzer for his valuable advice.

REFERENCES

1. P. L. BUTZER AND R. J. NESSEL, "Fourier Analysis and Approximation," Vol. 1, Academic Press, New York, 1971.
2. P. L. BUTZER AND H. J. WAGNER, Walsh–Fourier series and the concept of a derivative, *Appl. Anal.* **3** (1973), 29–46.
3. W.-Y. SU, The kernel of Abel–Poisson type on Walsh system, *Chinese Ann. Math.* **2** (1981), 81–92. [English]
4. W.-Y. SU AND W.-X. ZHENG, The logical derivatives and integrals, *J. Math. Res. Exposition* **1** (1981), 79–90. [Chinese]
5. W.-X. ZHENG, The generalized Walsh transform and an extremum problem, *Acta Math. Sinica* **22**, No. 3 (1979), 362–374. [Chinese]
6. W.-X. ZHENG, Approximate identity kernels on Walsh system, *Chinese Ann. Math. Ser. A* **4** (1983), 177–184. [Chinese]